Final exam (duration: 2hr)
Financial maths and statistics

Check that this exam paper has 2 pages. This is a closed-book exam. No materials are permitted. Non-programmable calculators are allowed. All other computing, communicating, and electronic devices must be switched off and placed in your bag/backpack at the front of the examination room.

Exercises are independent from each other. Stick to the notation in the exercise if one is provided. Points will be awarded based on the solution method (formulas used, calculations) not merely on the final answer. Report all results rounded to two decimal places (4 for percentage rates) while intermediate computations should be carried out with at least 4 significant figures.

Exercise 1

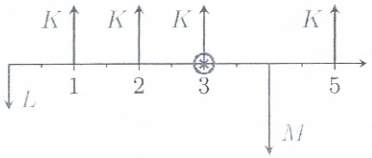
1. A savings account is advertised to pay an annual percentage rate of 4 %, compounded semiannually.
   a. What is the effective annual rate?
   b. What is the effective interest rate over a month?
   c. What is the mortgage equivalent yield (MEY, annualized yield with monthly compounding)?
   d. What is the equivalent interest rate, compounded twice a month?

For each of the following questions, start with a cash flow diagram.

2. How much would you get after 10 years if you deposit 1000 € every quarter on this account?

3. How much should you pay every year starting at the end of the year for 3 years if you want to be able to withdraw 15 000 € in one year and a half and 7500 € in 4 years from now?

4. Write the equation that relates the cash inflows to the cash outflows at the starred time in the following diagram:



5. How long would it take to get 100 000 € on this account if you can start by depositing 2500 € at the end of the year and then keep depositing every 6 months from then on at a growth rate of 2 %?

Exercise 2

1. Starting from the present value of a standard annuity, give the expression of its future value (at the time of the last cash flow). How about growing annuities?

Twenty-five years before retiring, Ben wants to build up a supplementary pension provision. He finds a pension savings account earning 4 % a year. No withdrawals are allowed on this account before retirement.

2. What is the equivalent interest rate, compounded monthly (r_m)? semiannually (r_s)?

Ben wants to be able to get an additional income of at least 6000 € twice a year for the first 12 years after he retires.

3. What is the value of his pension benefit upon retirement?

4. How much should his monthly contributions be to be able to meet his goals?

Often, contributions to pension savings accounts are a fraction f of income. At the moment of his first deposit, the salary of Ben is 2000 €. His salary will increase at a rate of 0.25 % every month until he retires.

5. What fixed percentage of his income (f) should he pay each month to meet his goals?

Exercise 3

An asset manager estimates that three scenarios are likely to occur at a horizon of one year: an economic expansion (E), a stable economy (S), and an economic downturn (D). She thinks that the three assets she considers investing in will behave as in the following tables:

Asset	Value today (£)	Value in one year (£)		
		E	S	D
F	32	36	34	30
G				62.1
H	50		52	

Probability of occurrence (%)	35	40	25
	E	S	D
Return on F (%)			
Return on G (%)	6.0	−4.0	3.5
Return on H (%)	8.0		−5.0

1. Many data are missing so reproduce these two tables and help her fill in the blanks.

2. Compute the expected return and the variance for each asset.

3. Compute the variance-covariance matrix of these asset returns.

4. What is the correlation between assets G and H?

5. Establish the general expression of the expected return and variance of a portfolio made only with two assets A and B with weights w_A and w_B in terms of the expected return, the volatility of each component return, and their correlation ρ_{AB}.

6. Deduce the weights of the minimum variance portfolio that can be made out of assets A and B (Hint: remember that $w_B = 1 - w_A$).

7. What are the characteristics of the minimum variance portfolio V given assets G and H (weights of each component asset, expected return and volatility)?

8. How should the asset manager allocate 8 000 000 £ in order to get the minimum variance portfolio?

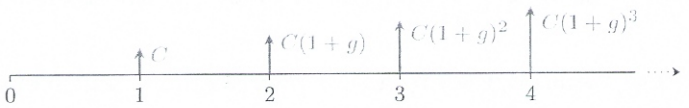
Formulas

- The present value of a standard annuity of n cash flows, all with the same amount A, is given by

$$\text{PV} = \sum_{t=1}^{n} \frac{A}{(1+r)^t} = \frac{A}{r}\left(1 - \frac{1}{(1+r)^n}\right)$$

  where r is the effective interest rate over one period.
- The present value of a growing annuity with n cash flows is

$$\text{PV} = \frac{C}{r-g}\left(1 - \left(\frac{1+g}{1+r}\right)^n\right)$$



- If the IRR is between r_1 and r_2, a linear interpolation allows to get an approximate value:

$$\frac{r_1 - \text{IRR}}{r_1 - r_2} = \frac{\text{NPV}_1 - 0}{\text{NPV}_1 - \text{NPV}_2} \Leftrightarrow \text{IRR} = r_1 + \frac{(r_2 - r_1)}{(\text{NPV}_1 - \text{NPV}_2)} \times \text{NPV}_1$$

- The expected return of an asset indexed by i is given by $E(r_i) = \overline{r_i} = \mu_i$. The variance is denoted by $\text{Var}(r_i) = \sigma_i^2 = E(r_i^2) - \mu_i^2$.
- The covariance between returns r_i and r_j is defined as $\text{Cov}(r_i, r_j) = \sigma_{ij} = E(r_i r_j) - \mu_i \mu_j$ while their correlation is $\rho_{ij} = \dfrac{\sigma_{ij}}{\sigma_i \sigma_j}$.